

The Sound of Symmetry

Zhiqin Lu, [University of California, Irvine](#)

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(Can one hear the shape of a drum?)

For a domain Ω in \mathbb{R}^2 , the above mentioned *spectrum* is the set of eigenvalues of the Laplace operator Δ with Dirichlet boundary condition. This is the set of all real numbers λ such that there exists a smooth function u on $\bar{\Omega}$ satisfying the Laplace equation and boundary condition

$$\Delta u(x, y) := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda u(x, y),$$

$u = 0$ on the boundary of Ω .

The eigenvalues of a bounded domain form a discrete set of $(0, \infty)$

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \dots$$

The set of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ is in bijection with the resonant frequencies a drum would produce if Ω were its drumhead. With a perfect ear one could hear all these frequencies and therefore know the spectrum. In other words, if two drums sound identical to a perfect ear and therefore have identical spectra, then are they the same shape?

Let's take a look at the simplest case and assume that our drum is actually a string which is mathematically described by the ordinary differential equation

$$f''(x) = -\lambda f(x), \quad f(0) = f(\ell) = 0.$$

We know from calculus that the solutions are

$$f_k(x) = \sin\left(\frac{k\pi x}{\ell}\right), \quad \text{with } \lambda = \lambda_k = \frac{k^2\pi^2}{\ell^2}$$

for $k = 1, 2, \dots$.

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Can one hear the shape of a string?

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$$\ell = \sqrt{\frac{\pi^2}{\lambda_1}}.$$

This shows that we can hear the length of the string based on the first eigenvalue.

Exercise

Prove that one can hear the shape of a rectangle.

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Mathematically, the question is equivalent to determining whether or not the following map

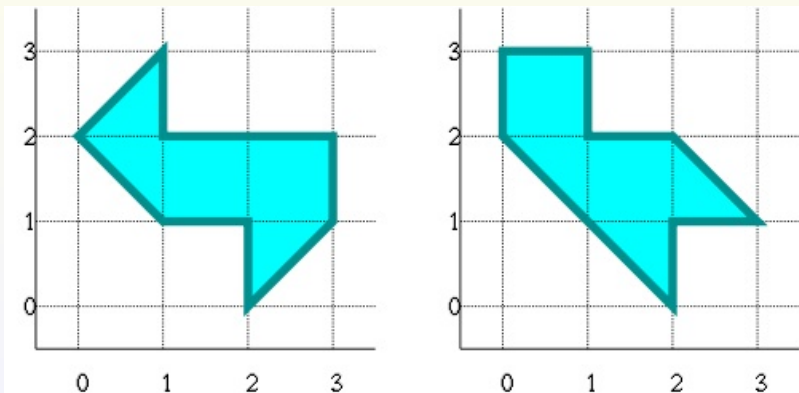
$$\Lambda : \mathcal{M} \rightarrow \mathbb{R}^\infty, \quad \Omega \mapsto (\lambda_1, \lambda_2, \dots, \lambda_k, \dots),$$

is injective, where \mathcal{M} is the moduli space of all bounded domains in \mathbb{R}^2 .

Counterexample of Kac's question

Gordon, Webb, and Wolpert gave a counterexample:

Figure : Identical sounding drums



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- 2 Motivated by these problems we shall investigate the injectivity of Λ restricted to certain subsets of \mathcal{M} .
- 3 A natural choice is the set of convex n -gons since this set can be identified with a finite dimensional manifold with corners.
- 4 Surprisingly, even within this subset Kac's question is a subtle problem. Durso (1988) proved that one *can* hear the shape of a triangle, so in fact Λ restricted to the moduli space of Euclidean triangles is injective. For $n > 3$ the problem is widely open.

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In particular, $\lambda_k \rightarrow \infty$ and Δ is an unbounded operator.

How to prove that

$$\sum_{k=1}^{\infty} e^{-\lambda_k t}$$

is finite for any $t > 0$?

The heat kernel is a smooth function $H(x, y, t)$ such that

$$\begin{cases} \partial_t H(x, y, t) = \Delta_x H(x, y, t) \\ H(x, y, t) = 0 \quad \text{for } x \text{ on the boundary} \\ H(x, y, 0) = \delta_x(y) \quad \text{is the dirac function} \end{cases}$$

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$$\sum_{k=1}^{\infty} e^{-\lambda_k t} = \text{Tr}(e^{\Delta t}) = \int_{\Omega} H(x, x, t) dx.$$

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$$\frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}}.$$

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By **our life experience**, $H(x, y, t)$ is very small when $d(x, y)$ is big or t is small!

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If x is in the interior of Ω , then for y close enough, then

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Let Ω be the n -polygon.

If x is in the interior of Ω , then for y close enough, then

$$H(x, y, t) \approx \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}}.$$

If x is near the edge (assuming that the edge is $x_2 = 0$), the above computation should have a correction term by the Schwarz reflexion principle:

$$H(x, y, t) \approx \frac{1}{4\pi t} \left(e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x'-y|^2}{4t}} \right), \quad x = (x_1, -x_2)$$

Thus we have

$$H(x, x, t) = \frac{1}{4\pi t} \quad \text{if } x \text{ is in the interior}$$

$$H(x, x, t) = \frac{1}{4\pi t} \left(1 - e^{-\frac{x^2}{t}}\right) \quad \text{if } x \text{ is near the boundary}$$

$$\begin{aligned}\sum e^{-\lambda_k t} &\approx \int_{\Omega} \frac{1}{4\pi t} dx + \int_{\partial\Omega_\varepsilon} \frac{1}{4\pi t} e^{-\frac{x_2^2}{t}} dx_1 dx_2 \\ &\approx \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}}\end{aligned}$$

as $t \rightarrow 0$.

A more accurate result

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{i=1}^n \frac{\pi^2 - \alpha_i^2}{24\pi\alpha_i} + O(e^{-1/t}) \quad t \downarrow 0.$$

This shows that the area, the perimeter, and the sum over the angles of $\frac{\pi^2 - \alpha^2}{24\pi\alpha}$ are all spectral invariants.

Let

$$k(x, y, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} - \left(\frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{i=1}^n \frac{\pi^2 - \alpha_i^2}{24\pi\alpha_i} \right).$$

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Conjecture (Lu)

There exists a number $\beta > 0$ such that for any $\varepsilon > 0$ we have

$$C(\varepsilon)e^{-\frac{\beta+\varepsilon}{t}} \leq k(x, y, t) \leq C(\varepsilon)^{-1}e^{-\frac{\beta-\varepsilon}{t}}$$

for a constant $C(\varepsilon)$.

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At the current technology, we can only make use of three equations. More spectrum invariants?

The spectrum also determines the *wave trace*. Imagine a convex n -gon is a billiard table. The set of closed geodesics is precisely the set of all paths along which a billiard ball hit with a pool cue could roll, such that the ball returns to its starting point. The set of lengths of closed geodesics, which is known as *the length spectrum*, is related to the (Laplace) spectrum by a deep result proven by Duistermaat and Guillemin in the late 1970s.

The wave trace is a tempered distribution defined by

$$\sum_{k=1}^{\infty} e^{i\sqrt{\lambda_k}t}.$$

Duistermaat and Guillemin proved in that the singularities of the wave trace is contained in the length spectrum.

Example (The wave trace of a string of length π)

$$\sum_{k=0}^{\infty} e^{ikt} = \frac{1}{1 - e^{it}}.$$

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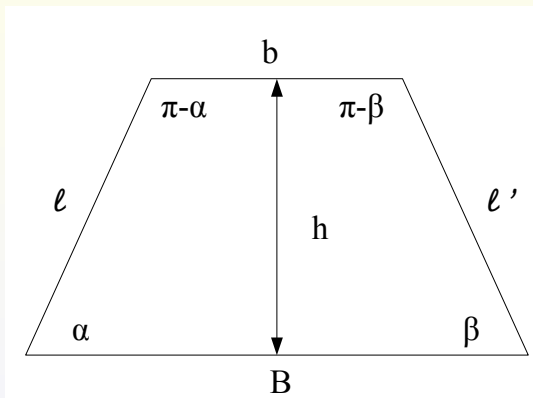
4 dimensional moduli space!

Answer: partially yes, by L-Rowlett!

Theorem (L-Rowlett-2013)

If two acute trapezoids have the same spectrum, then they are identical up to rigid motions of the plane.

Figure : An acute trapezoid



Lemma

The length of the shortest closed geodesic of an acute trapezoid is twice the height, that is twice the distance from the base to the opposite parallel side.

The four equations

$$\left\{ \begin{array}{l} h = h_1 \\ \frac{1}{2}(B + b)h = \frac{1}{2}(B_1 + b_1)h_1 \\ \ell + \ell' + B + b = \ell_1 + \ell'_1 + B_1 + b_1 \\ \frac{1}{\alpha(\pi - \alpha)} + \frac{1}{\beta(\pi - \beta)} = \frac{1}{\alpha_1(\pi - \alpha_1)} + \frac{1}{\beta_1(\pi - \beta_1)} \end{array} \right.$$

The deduction

$$h(\csc \alpha + \csc \beta) = \ell + \ell' = L - 2A/h$$

where L is the perimeter and A is the area.

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$$\csc \alpha + \csc \beta = \csc \alpha_1 + \csc \beta_1.$$

We have

$$\begin{cases} \csc \alpha + \csc \beta = \csc \alpha_1 + \csc \beta_1 \\ \frac{1}{\alpha(\pi-\alpha)} + \frac{1}{\beta(\pi-\beta)} = \frac{1}{\alpha_1(\pi-\alpha_1)} + \frac{1}{\beta_1(\pi-\beta_1)} \end{cases}$$

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Lemma (A Putnam-type problem)

We have

$$\alpha = \alpha_1, \beta = \beta_1$$

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solved by L-Rowlett, 2013

Theorem

Let p, q be real numbers. Then the solution of the system of equations

$$\begin{cases} \csc(\alpha) + \csc(\beta) = p; \\ (\alpha(\pi - \alpha))^{-1} + (\beta(\pi - \beta))^{-1} = q, \end{cases}$$

if it exists, must be unique for $0 < \beta \leq \alpha \leq \pi/2$.

First, let's use the second equation to show that each α uniquely determine a β . Solving the second equation for β in terms of α and q leads to a quadratic equation in β ,

$$\beta^2(1 - q\alpha(\pi - \alpha)) + \beta(\pi(q\alpha(\pi - \alpha) - 1)) - \alpha(\pi - \alpha) = 0.$$

Since the trapezoid is acute, $\beta \leq \frac{\pi}{2}$, so indeed α and q uniquely determine

$$\beta = \beta(\alpha) = \frac{\pi}{2} - \sqrt{\frac{\pi^2}{4} + \frac{\alpha(\pi - \alpha)}{1 - q\alpha(\pi - \alpha)}}.$$

We can prove the lemma if we prove that the function

$$g(\alpha) := \csc(\alpha) + \csc(\beta(\alpha))$$

has unique solution α for any given p . Implicit differentiation gives

$$\beta'(\alpha) = -\frac{((\alpha(\pi - \alpha))^{-1})'}{((\beta(\pi - \beta))^{-1})'},$$

and so the derivative

$$g'(\alpha) = -\csc(\alpha) \cot(\alpha) + \csc(\beta(\alpha)) \cot(\beta(\alpha)) \cdot \frac{((\alpha(\pi - \alpha))^{-1})'}{((\beta(\pi - \beta))^{-1})'}.$$

Let's see if we can relate $g'(\alpha)$ to the following function

$$f(\alpha) := \frac{\csc(\alpha) \cot(\alpha)}{((\alpha(\pi - \alpha))^{-1})'} = \frac{\alpha^2(\pi - \alpha)^2 \cos \alpha}{(2\alpha - \pi) \sin^2 \alpha}.$$

Since

$$\frac{g'(\alpha)}{((\alpha(\pi - \alpha))^{-1})'} = -\frac{\csc(\alpha) \cot(\alpha)}{((\alpha(\pi - \alpha))^{-1})'} + \frac{\csc(\beta(\alpha)) \cot(\beta(\alpha))}{((\beta(\pi - \beta))^{-1})'},$$

we see that

$$g'(\alpha) = \frac{\pi - 2\alpha}{\alpha^2(\pi - \alpha)^2} (f(\alpha) - f(\beta))$$

It turns out that the logarithmic derivative of f is pleasantly simple

$$\frac{f'(\alpha)}{f(\alpha)} = \frac{2}{\pi - 2\alpha} + \frac{2}{\alpha} + \frac{2}{\alpha - \pi} - 2 \cot(\alpha) - \tan(\alpha).$$

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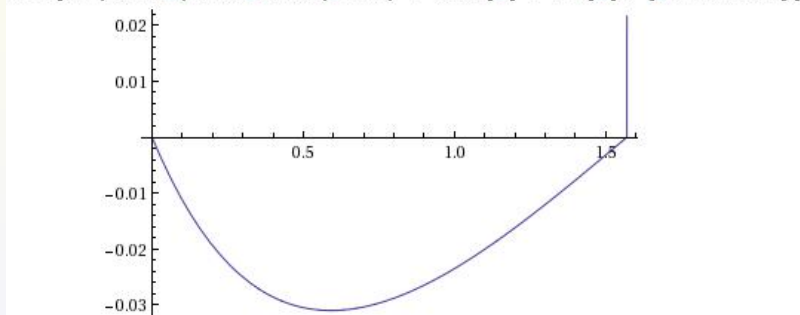
$$\frac{f'(\alpha)}{f(\alpha)} = \frac{2}{\pi - 2\alpha} + \frac{2}{\alpha} + \frac{2}{\alpha - \pi} - 2 \cot(\alpha) - \tan(\alpha).$$

Need to prove that the above expression is negative for $\alpha \in (0, \pi/2)$.

The use of MatLab

Figure : Graph of $(\log f)'$

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Plot[2 / (Pi - 2 x) + 2 / x + 2 / (x - Pi) - 2 Cot[x] - Tan[x], {x, 0, Pi / 2}]
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Thank you!