On a common generalization of the Böttcher-Wenzel Inequality and the Normal Scalar Curvature Inequality

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Dedicate to the memory of Franki Dillen
We would like to study some linear algebraic inequalities rising from submanifold theory.
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2. some linear algebraic inequalities related to the submanifold theory (I thank Bogdan Suceavă and David Wenzel for discussions)
3. open questions
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Let $M^n$ be a compact minimal submanifold of $S^{n+p}$.

1. Let $h^\alpha_{ij}$ be the second fundamental form (wrt orthonormal frames)
2. Let $A_\alpha = (h^\alpha_{ij})$
3. Then we have

$$\Delta A_\alpha = nA_\alpha - \langle A_\alpha, A_\beta \rangle A_\beta - [A_\beta, [A_\beta, A_\alpha]].$$
\[ \frac{1}{2} \Delta \| \sigma \|^2 = \sum_{i,j,k} (h_{ijk}^\alpha)^2 + n \| \sigma \|^2 - \sum_\alpha,\beta \| [A_\alpha, A_\beta] \|^2 - \sum_\alpha,\beta |\langle A_\alpha, A_\beta \rangle|^2. \]
\[
\frac{1}{2} \Delta \|\sigma\|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n \|\sigma\|^2 - \sum_{\alpha,\beta} \| [A_\alpha, A_\beta] \|^2 - \sum_{\alpha,\beta} |\langle A_\alpha, A_\beta \rangle|^2.
\]

By the Chern-do Carmo-Kobayashi inequality below, we have

\[
\frac{1}{2} \Delta \|\sigma\|^2 \geq n \|\sigma\|^2 - 2 \sum_{\alpha \neq \beta} \|A_\alpha\|^2 \|A_\beta\|^2 - \sum_{\alpha} \|A_\alpha\|^4
\]

\[
\geq n \|\sigma\|^2 - (2 - \frac{1}{p}) \|\sigma\|^4.
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\geq n\|\sigma\|^2 - (2 - \frac{1}{p})\|\sigma\|^4.
\]

If \(\|\sigma\|\) is a constant, then we have either \(\|\sigma\| = 0\), or

\[
\|\sigma\|^2 \geq \frac{n}{(2 - p^{-1})}.
\]
Consider minimal submanifolds of sphere

This is called the gap phenomena.

1. J. Simons proved a gap theorem for the length of the second fundamental form of minimal submanifold in sphere

2. Generalizations of J. Simons result was obtained by Chern-do Carmo-Kobayashi, Yau, Shen, Wu-Song, Li-Li, Chen-Xu

3. We have a new method
Some examples of minimal submanifolds:
Some examples of minimal submanifolds:

1. Totally geodesic minimal submanifolds; \( \| \sigma \|^2 = 0 \)

2. \( M_{r,n-r} = S^r \left( \sqrt{\frac{r}{n}} \right) \times S^{n-r} \left( \sqrt{\frac{n-r}{n}} \right); \| \sigma \|^2 = n \)

3. The Veronese surface is defined as the immersion \( S^2 \to S^4; \| \sigma \|^2 = \frac{4}{3} = \frac{2}{3}n \)

\[
\begin{align*}
    u^1 &= \frac{1}{\sqrt{3}}yz, \\
    u^2 &= \frac{1}{\sqrt{3}}zx, \\
    u^3 &= \frac{1}{\sqrt{3}}xy, \\
    u^4 &= \frac{1}{2\sqrt{3}}(x^2 - y^2), \\
    u^5 &= \frac{1}{6}(x^2 + y^2 - 2z^2).
\end{align*}
\]

\((x, y, z) \in S^2\).
The optimal results are
The optimal results are

1. For minimal hypersurface of dim $n$, if $0 \leq ||\sigma||^2 \leq n$, then either $||\sigma|| = 0$ or $||\sigma||^2 = n$ (well known, Chern-do Carmo-Kobayashi);

2. For high co-dimensional minimal submanifolds, if $0 \leq ||\sigma||^2 \leq \frac{2}{3}n$, then they are totally geodesic or Veronese surface. (Li-Li, Chen-Xu)
The new method

Taking $\Delta$ on $\max \|A_r\|^2$ instead of $\|\sigma\|^2 = \sum_r \|A_r\|^2$.

New J. Simons-Bochner formula (Lu-2007):

$$\frac{1}{2} p \Delta f_p = \frac{1}{2} \sum_{s, t = p - 2} \lambda_s \lambda_t \left( D \partial_x^2 \partial_x a_{\alpha\beta} \right) + \sum_{\alpha} \left( \lambda_{p - 1} \alpha \sum_{i, j, k} (h_{\alpha ijk})^2 \right) + nf_p - f_{p + 1} - \sum_{\beta \neq \alpha} \| [A_{\beta}, A_{\alpha}] \|_2 \| A_{\alpha} \|_{p - 1},$$

where $f_p = \text{tr} (F_p)$, $F_p = \langle A_i, A_j \rangle$.

$\lim f_p = \max \| A_r \|^2$. Zhiqin Lu UC Irvine

On a common generalization
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$$\frac{1}{2p} \Delta f_p = \frac{1}{2} \sum_{s+t=p-2} \sum_{k,\alpha,\beta} \lambda_s^s \lambda_t^t \left( D \frac{\partial}{\partial x_k} a_{\alpha\beta} \right)^2 + \sum_{\alpha} \left( \lambda_{\alpha}^{p-1} \sum_{i,j,k} (h_{ijk}^\alpha)^2 \right)$$

$$+ nf_p - f_{p+1} - \sum_{\beta \neq \alpha} \| [A^\beta, A^\alpha] \|^2 \|A^\alpha\|^{p-1},$$

where $f_p = \text{tr}(F^p)$, $F = \langle A_i, A_j \rangle$, $f_1 = \|\sigma\|^2$.

$$\lim \frac{1}{f_p} = \max \|A_r\|. $$
The technical heart of the original J. Simons gap theorem is the following result of Chern-do Carmo-Kobayashi (in 70's)
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**Theorem**

If $A, B$ are symmetric matrices, then

$$\|[A, B]\|^2 \leq 2\|A\|^2\|B\|^2$$

Since the inequality is invariant under orthogonal transformation, we can assume that $A$ is diagonalized.

Then the inequality becomes

$$\sum (\lambda_i - \lambda_j)^2 b_{ij}^2 \leq 2\|A\|^2\|B\|^2.$$

$$(\lambda_i - \lambda_j)^2 \leq 2(\sum \lambda_k^2) = 2\|A\|^2$$
The technical heart of the new gap theorem is the following

**Theorem (Lu-2007)**

Let $A$ be a real symmetric matrix of norm 1. Let $A_1, \cdots, A_m$ be symmetric matrices such that

1. $\langle A_\alpha, A_\beta \rangle = 0$ if $\alpha \neq \beta$;
2. $\|A_1\| \geq \cdots \geq \|A_m\|$.

Then we have

$$\sum_{\alpha=1}^{m} \| [A, A_\alpha] \|^2 \leq \sum_{\alpha=1}^{m} \| A_\alpha \|^2 + \| A_1 \|^2.$$
The equality holds iff, after an orthonormal base change and up to a sign, we have

- $A_2 = \cdots = A_m = 0$, and

$$A = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 \\
0 & -\frac{1}{\sqrt{2}} \\
0 & \ddots & \ddots \\
\end{pmatrix}$$

$$A_1 = c \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 \\
\ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots \\
\end{pmatrix},$$
Or

For two real numbers $\lambda = 1/\sqrt{n(n-1)}$ and $\mu$, we have

$$A = \lambda \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & -1 \end{pmatrix},$$

and $A_\alpha$ is $\mu$ times the matrix whose only nonzero entries are 1 at the $(1, \beta)$ and $(\beta, 1)$ places, where $\beta = 2, \cdots, n$. 

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On a common generalization
We have new gap theorems that generalize all the previous results.
Consider a surface in $R^3$. Principal curvature at $P \in S : k_1, k_2$

2 Gaussian curvature: $K(p) = k_1 \cdot k_2$

3 Mean curvature: $H(p) = \frac{k_1 + k_2}{2}$

4 Then we have

$$K(p) \leq H^2(p) \quad \text{Intrinsic} \leq \text{Extrinsic}$$
Let $M^n$ be an $n$-dimensional manifold isometrically immersed into the space form $N^{n+m}(c)$ of constant sectional curvature $c$. 

Then we have $\rho \leq |H|^2 + c$ where $H$ is the mean curvature tensor and $\rho$ is the normalized scalar curvature $\rho = \frac{1}{n} \left( \frac{n}{n-1} \sum R_{ijij} \right)$. 
Let $M^n$ be an $n$-dimensional manifold isometrically immersed into the space form $N^{n+m}(c)$ of constant sectional curvature $c$. Then we have

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where $H$ is the mean curvature tensor and $\rho$ is the normalized scalar curvature

$$\rho = \frac{1}{n(n-1)} \sum_{ij} R_{ijij}.$$
Theorem (B. Y. Chen, 1993)

Let $M^n$ be a submanifold in a space form of constant sectional curvature $c$. Then

$$n(n-1)\rho - \inf(\sec) \leq \frac{n^2(n-2)}{2(n-1)}|H|^2 + \frac{(n+1)(n-2)}{2}c.$$  

The equality case is completely determined by the form of the shape operators with respect to a suitable o.n. frame fields.
Theorem (B. Y. Chen, 1993)

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The equality case is completely determined by the form of the shape operators with respect to a suitable o.n. frame fields.

Define the curvature invariant $\delta = n(n-1)\rho - \inf(\sec)$. Then it is an intrinsic obstruction to the immersion to an ambient space.
Let $M^n$ be an immersed submanifold of the space form $N^{n+m}(c)$. The (normalized) scalar curvature of the normal bundle is defined by De Smet, Dillen, Verstraelen, and Vrancken as:

$$
\rho_\perp = \frac{1}{n(n-1)}|R_\perp|,
$$

where $R_\perp$ is the curvature tensor of the normal bundle.
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$$\rho^\perp = \frac{1}{n(n-1)} |R^\perp|, $$

where $R^\perp$ is the curvature tensor of the normal bundle. More precisely, let $\xi_1, \cdots, \xi_m$ be a local orthonormal frame of the normal bundle. Then

$$\rho^\perp = \frac{2}{n(n-1)} \left( \sum_{1=i<j}^{n} \sum_{1=r<s}^{m} \langle R^\perp(e_i, e_j)\xi_r, \xi_s \rangle^2 \right)^{\frac{1}{2}}.$$
\( \rho^\perp \) is always nonnegative.
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**Conjecture**

Let \( \sigma \) be the second fundamental form of \( M \), and let \( H = \frac{1}{n} \text{trace} \sigma \) be the mean curvature tensor. Then

\[
\rho + \rho \perp \leq |H|^2 + c.
\]
1. Partial results were obtained by Chern-do Carmo-Kobayashi, Rouxel, Guadalupe-Rodriguez, De Smet-Dillen-Verstraelen-Vrancken, Choi-Lu, Dillen-Fastenakels-Veken, etc.

2. I proved the conjecture in 2007.

3. Ge-Tang gave an independent proof slightly later.
The algebraic version of the Normal Scalar Curvature Conjecture can be stated as follows:

\[\text{Theorem}\]

Let \(A_1, \ldots, A_m\) be symmetric matrices. For \(n, m \geq 2\), we have

\[2(\sum_{r<s} \| [A_r, A_s] \|^2) \leq (\sum_{r=1}^m \| A_r \|^2)^2\]

It is interesting that

\[m \sum_{\alpha=2} \| [A_1, A_{\alpha}] \|^2 \leq m \sum_{\alpha=2} \| A_{\alpha} \|^2 + \| A_1 \|^2\]

implies the above.

Proof: let \(t = \| A_1 \|^2\) and write the DDVV inequality as a quadratic inequality of \(t\).
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On a common generalization
We seek a geometric interpretation of our fundamental formula. Let

$$\text{Ric}^\perp(e_\alpha, e_\alpha) = \sqrt{\sum_{\beta \neq \alpha} |R^\perp_{\alpha\beta}|^2}$$

Then we have

$$|\text{Ric}^\perp(e_\alpha, e_\alpha)|^2 \leq \|B_\alpha\|^2(-n(n-1)(\rho-|H|^2-c)-\|B_\alpha\|^2+\max_{\gamma \neq \alpha} \|B_\gamma\|^2),$$

where $B_\alpha$ is the traceless part of $A_\alpha$, the second fundamental form in the direction $e_\alpha$. 
We call the above inequality *Normal Ricci curvature inequality*. Although the proof is not straightforward, we have

**Theorem**

*The Normal Ricci curvature inequality implies the Normal scalar curvature inequality.*
There are two problems we can ask about the equality case of the Normal Scalar Curvature Conjecture.

1. If the DDVV inequality is true at one point, what are the shape matrices $A_r$? This problem has completely been solved since we solved the equality case of the sharper inequality
\[ m \sum_{\alpha=2}^{\alpha=2} \| [A_1, A_{\alpha}] \|_2 \leq m \sum_{\alpha=2}^{\alpha=2} \| A_{\alpha} \|_2 + \| A_1 \|_2. \]

2. Classify all the submanifolds for which the DDVV inequality is an equality at any point of the manifold. This is a much more interesting question and hasn't been completely solved yet. See Choi-Lu-2008 for details.

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The equality case

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Definition

Let $M^n \to N^{n+m}(c)$ be an immersed submanifold of the space form. Let $\sigma$ be the second fundamental form of the submanifold. Let $\nu$ be any normal vector of the submanifold. $M^n$ is called \textit{austere}, if for any $r$ and any $0 \leq k < n/2$, we have $\sigma_{2k+1}(A_r) = 0$, where $\sigma_k(A)$ is the $k$-th elementary polynomial of the matrix $A$. 

Corollary

Assume that $M$ is 3 dimensional and minimal. If the equality of the DDVV is valid at any point and $M$ is minimal, then $M$ is an \textit{austere} 3-fold.
Definition

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**Theorem**

Let $A_1, \ldots, A_m$ be symmetric matrices. For $n, m \geq 2$, we have

$$\left(\sum_{r=1}^{m} \|A_r\|^2\right)^2 \geq 2\left(\sum_{r<s} \|[A_r, A_s]\|^2\right).$$
Erdős-Mordell inequality (1935)—an application

\[ PA + PB + PC \geq 2(PP_A + PP_B + PP_C). \]
Erdös-Mordell inequality is the dual version of the Normal Scalar Curvature Conjecture!
Erdös-Mordell inequality is the dual version of the Normal Scalar Curvature Conjecture!

1. Is $a^2 + b^2 + c^2 \geq ab + bc + ca$ a generalization of $a^2 + b^2 \geq 2ab$?

2. Not really!

3. The correct generalization is: if $\alpha + \beta + \gamma = \pi$, then

$$a^2 + b^2 + c^2 \geq 2ab \cos \gamma + 2bc \cos \alpha + 2ac \cos \beta,$$

which implies the Erdös-Mordell inequality.
We consider vectors $\vec{x}, \vec{y}, \vec{z}$ in $\mathbb{R}^2$ such that the angles between them are $\pi/2 + \gamma/2, \pi/2 + \beta/2, \pi/2 + \alpha/2$, respectively. Moreover, we assume that that norm of $\vec{x}, \vec{y}, \vec{z}$ are $a, b, c$.

Let $X, Y, Z$ be the corresponding $3 \times 3$ skew-symmetric matrices.

Then the Erdős-Mordell inequality is equivalent to

$$\|[X, Y]\|^2 + \|[Y, Z]\|^2 + \|[Z, X]\|^2 \leq \frac{1}{8} (\|X\|^2 + \|Y\|^2 + \|Z\|^2)^2.$$
Go back to the Bochner formula. We have studied the Bochner formula of $\|\sigma\|^2$ and $\max\|A_r\|^2$. Can we get more information by studying the Simons-Bochner-type formula for $\|A_r\|^2$ for any $r$?
Go back to the Bochner formula. We have studied the Bochner formula of $||σ||^2$ and $\max ||A_r||^2$. Can we get more information by studying the Simons-Bochner-type formula for $||A_r||^2$ for any $r$?

**Conjecture (Böttcher-Wenzel)**

Let $A, A_1$ be $n \times n$ real matrices. Then

$$||[A, A_1]||^2 \leq 2||A||^2||A_1||^2.$$
Go back to the Bochner formula. We have studied the Bochner formula of $\|\sigma\|^2$ and $\max\|A_r\|^2$. Can we get more information by studying the Simons-Bochner-type formula for $\|A_r\|^2$ for any $r$?

**Conjecture (Böttcher-Wenzel)**

Let $A, A_1$ be $n \times n$ real matrices. Then

$$\|[A, A_1]\|^2 \leq 2\|A\|^2\|A_1\|^2.$$ 

This is a generalization of the Chern-do Carmo-Kobayashi inequality.
The conjecture was proved by Lu in 2007 (posted on arXiv on 11/22/2007); after that there are three different proofs by Vong-Jin, Böttcher-Wenzel, and Audenaert.

We sharpened the Schrödinger-Robertson relation, which was

\[ \|[A, B]\|^2 \leq 4\|A\|^2\|B\|^2. \]

The optimal constant is 2, as proved.
Equality case

Let

\[ A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \]

Then

\[ \| [A, A_1] \| = \sqrt{2} \| A \| \cdot \| A_1 \|. \]

Therefore, the result is optimal.
We make a remark that all the linear algebraic inequalities in this talk can be generalized to separable Hilbert spaces. It would be very interesting to give these inequalities purely functional analytic proofs.
In summary, we have the following results: let $A, A_1, \cdots, A_m$ be real matrices. Then

$$\| [A, A_1] \|^2 \leq 2 \| A \|^2 \| A_1 \|^2.$$
In summary, we have the following results: let $A, A_1, \ldots, A_m$ be real matrices. Then

$$
\|[A, A_1]\|^2 \leq 2\|A\|^2\|A_1\|^2.
$$

If all $A, A_1, \ldots, A_m$ are symmetric, then

$$
\sum_{\alpha=1}^{m} \|[A, A_\alpha]\|^2 \leq \|A\|^2(\sum_{\alpha=1}^{m} \|A_\alpha\|^2 + \max_{1 \leq r \leq m} \|A_r\|^2).
$$
In summary, we have the following results: let $A, A_1, \cdots, A_m$ be real matrices. Then

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If all $A, A_1, \cdots, A_m$ are symmetric, then

$$\sum_{\alpha=1}^{m} \| [A, A_\alpha] \|^2 \leq \| A \|^2 \left( \sum_{\alpha=1}^{m} \| A_\alpha \|^2 + \max_{1 \leq r \leq m} \| A_r \|^2 \right).$$ 

Common generalization?
Consider

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Then

\[ \|[A, A_1]\|^2 + \|[A, A_2]\|^2 = 4 > 3 = \|A\|^2(\|A_1\|^2 + \|A_2\|^2 + \|A_1\|^2). \]
Let $V = \mathfrak{gl}(n, \mathbb{R})$ and define a linear map $T = T_X$ by

$$T : V \to V, \quad Y \mapsto [X^T, [X, Y]],$$

where $X^T$ is the transpose of $X$.

**Lemma**

$T$ is a symmetric operator, moreover, if $Y$ is an eigenvector of $T$, so is $[X, Y]^T$, provided that it is not zero.
Using the above setting, the Böttcher-Wenzel inequality can be written as

$$\lambda_1(T) \leq 2,$$

where $\lambda_1$ is the largest eigenvalue of $T$. 
What should we expect if the number of matrices is more than 2? Since the non-zero eigenvalue of $T$ are in pairs, we want, for example, that

$$A_2 \perp A_1$$

Based on the above observation, we make Conjecture (Fundamental Conjecture)

Assume that

$$\text{Tr}(A_{[A_i, A_j]}) = 0$$

for any $i, j$. Then

$$\sum_{\alpha=1}^{m} \|A_{[A_\alpha]}\|_2 \leq \|A\|_2 \left( \sum_{\alpha=1}^{m} \|A_\alpha\|_2 + \max_{1 \leq r \leq m} \|A_r\|_2 \right).$$

is true.
What should we expect if the number of matrices is more than 2? Since the non-zero eigenvalue of $T$ are in pairs, we want, for example, that

1. $A_2 \perp A_1$
2. $A_2 \perp [A_1, A]^T$.

Based on the above observation, we make Conjecture (Fundamental Conjecture)

Assume that $\text{Tr}(A_i A_j) = 0$ for any $i, j$. Then

$$m \sum_{\alpha=1} \|A_\alpha\|_2 \leq \|A\|_2 (m \sum_{\alpha=1} \|A_\alpha\|_2 + \max_{1 \leq r \leq m} \|A_r\|_2).$$

is true.

Zhiqin Lu UC Irvine On a common generalization
What should we expect if the number of matrices is more than 2? Since the non-zero eigenvalue of $T$ are in pairs, we want, for example, that

1. $A_2 \perp A_1$
2. $A_2 \perp [A_1, A]^T$.

Based on the above observation, we make

**Conjecture (Fundamental Conjecture)**

Assume that

$$\text{Tr} (A [A_i, A_j]) = 0$$

for any $i, j$. Then

$$\sum_{\alpha=1}^{m} \|[A, A_\alpha]\|^2 \leq \|A\|^2 (\sum_{\alpha=1}^{m} \|A_\alpha\|^2 + \max_{1 \leq r \leq m} \|A_r\|^2).$$

is true.
For the DDVV version, we make

**Conjecture**

Assume that $A_1, \cdots, A_m$ satisfy the cycle conditions

$$\text{Tr} \left( A_\alpha A_\beta A_\gamma \right) = \text{Tr} \left( A_\alpha A_\gamma A_\beta \right)$$

for any $1 \leq \alpha, \beta, \gamma \leq m$. Then we have

$$2 \left( \sum_{\alpha < \beta} \| [A_\alpha, A_\beta] \|^2 \right) \leq \left( \sum_{\alpha=1}^m \| A_\alpha \|^2 \right)^2.$$
Let $\lambda_1 = \lambda_2 \geq \lambda_3 = \lambda_4 \geq \cdots$ be the eigenvalues of $T$. Then we make

**Conjecture**

*If $\|A\| = 1$, then*

$$\lambda_1(T) + \lambda_3(T) \leq 3.$$
Let \( \lambda_1 = \lambda_2 \geq \lambda_3 = \lambda_4 \geq \cdots \) be the eigenvalues of \( T \). Then we make

**Conjecture**

*If \( \|A\| = 1 \), then*

\[
\lambda_1(T) + \lambda_3(T) \leq 3.
\]

**Lemma**

*The above conjecture is equivalent to our fundamental conjecture:*

\[
\sum_{\alpha=1}^{m} \|[A, A_{\alpha}]\|^2 \leq \|A\|^2 \left( \sum_{\alpha=1}^{m} \|A_{\alpha}\|^2 + \max_{1 \leq r \leq m} \|A_r\|^2 \right).
\]
Theorem

For $n = 3$, the fundamental conjecture is true.
Theorem

For $n = 3$, the fundamental conjecture is true.

Proof.

Without loss of generality, we may assume that $A$ is traceless. We compute

$$\text{Tr} (T) = 6 \|A\|^2 = 6$$

if $\|A\| = 1$.

Since nonzero eigenvalues of $T$ appear in pairs, we have

$$\lambda_1 + \lambda_3 \leq \frac{1}{2} \sum_{j=1}^{4} \lambda_j \leq \frac{1}{2} \text{Tr} (T) = 3.$$
The decomposition of $T$

Lemma

We have

$$T = \text{ad}^T \text{ad},$$

where $\text{ad}$ is the Lie derivative. Moreover, let

$$\text{ad}^* (X) = (\text{ad}(X))^T.$$

Then $\text{ad}^*$ is skew-symmetric, and

$$T = -(\text{ad}^*)^2.$$
The decomposition of $T$

**Lemma**

We have

$$T = \text{ad}^T \text{ad},$$

where $\text{ad}$ is the Lie derivative. Moreover, let

$$\text{ad}^*(X) = (\text{ad}(X))^T.$$

Then $\text{ad}^*$ is skew-symmetric, and

$$T = -(\text{ad}^*)^2.$$

So the spectrum of $T$ is more fundamental: it is related to the Lie algebra structure of the real special linear groups.
Open Questions

The conjectures in this talk.
What is the common generalization of the Normal Scalar Curvature Conjecture and Chen’s Fundamental inequality?
What is the upper bound of

\[ \sum_{j=0}^{k} \lambda_{1+2j}(T) \]

If \( k = 0 \), then the bound is 2 (Böttcher-Wenzel inequality), for \( k = 1 \), the bound is conjectured to be 3.
What are the corresponding inequalities on the Lie algebra other than the Lie algebra of the real special linear groups?
Open Questions

Classify all the submanifolds that the DDVV inequality is true at every point.
Open Questions

Are the inequalities in this talk true on any Hilbert spaces (not necessarily separable but with finite Hilbert-Schmidt norm)?
THANK YOU !